

# THE $(2k - 1)$ -CONNECTED MULTIGRAPHS WITH AT MOST $k - 1$ DISJOINT CYCLES

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**ABSTRACT.** In 1963, Corrádi and Hajnal proved that for all  $k \geq 1$  and  $n \geq 3k$ , every (simple) graph  $G$  on  $n$  vertices with minimum degree  $\delta(G) \geq 2k$  contains  $k$  disjoint cycles. The same year, Dirac described the 3-connected multigraphs not containing two disjoint cycles and asked the more general question: Which  $(2k - 1)$ -connected multigraphs do not contain  $k$  disjoint cycles? Recently, the authors characterized the simple graphs  $G$  with minimum degree  $\delta(G) \geq 2k - 1$  that do not contain  $k$  disjoint cycles. We use this result to answer Dirac's question in full.

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## 1. INTRODUCTION

For a multigraph  $G = (V, E)$ , let  $|G| = |V|$ ,  $\|G\| = |E|$ ,  $\delta(G)$  be the minimum degree of  $G$ , and  $\alpha(G)$  be the independence number of  $G$ . In this note, we allow multigraphs to have loops as well as multiple edges. For a simple graph  $G$ , let  $\overline{G}$  denote the complement of  $G$  and for disjoint graphs  $G$  and  $H$ , let  $G \vee H$  denote  $G \cup H$  together with all edges from  $V(G)$  to  $V(H)$ .

In 1963, Corrádi and Hajnal proved a conjecture of Erdős by showing the following:

**Theorem 1** ([1]). *Let  $k \in \mathbb{Z}^+$ . Every graph  $G$  with  $|G| \geq 3k$  and  $\delta(G) \geq 2k$  contains  $k$  disjoint cycles.*

The hypothesis  $\delta(G) \geq 2k$  is best possible, as shown by the  $3k$ -vertex graph  $H = \overline{K}_{k+1} \vee K_{2k-1}$ , which has  $\delta(H) = 2k - 1$  but does not contain  $k$  disjoint cycles. Recently, the authors refined Theorem 1 by characterizing all simple graphs that fulfill the weaker hypothesis  $\delta(G) \geq 2k - 1$  and contain  $k$  disjoint cycles. This refinement depends on an extremal graph  $Y_{k,k}$ .

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Let  $Y_{h,t} = \overline{K}_h \vee (K_t \cup K_t)$  (Figure 1.1), where  $V(\overline{K}_h) = X_0$  and the cliques have vertex sets  $X_1$  and  $X_2$ . In other words,  $V(Y_{h,t}) = X_0 \cup X_1 \cup X_2$  with  $|X_0| = h$  and  $|X_1| = |X_2| = t$ , and a pair  $xy$  is an edge in  $Y_{h,t}$  iff  $\{x, y\} \subseteq X_1$ , or  $\{x, y\} \subseteq X_2$ , or  $|\{x, y\} \cap X_0| = 1$ .

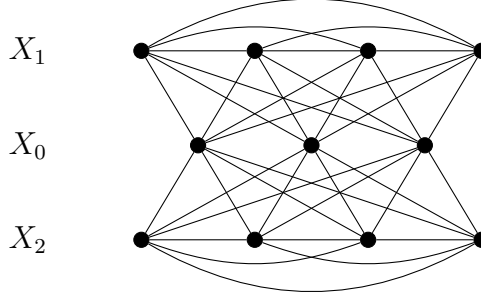


FIGURE 1.1.  $Y_{h,t}$ , shown with  $h = 3$  and  $t = 4$ .

**Theorem 2** ([4]). *Let  $k \geq 2$ . Every simple graph  $G$  with  $|G| \geq 3k$  and  $\delta(G) \geq 2k - 1$  contains  $k$  disjoint cycles if and only if:*

- (i)  $\alpha(G) \leq |G| - 2k$ ;
- (ii) if  $k$  is odd and  $|G| = 3k$ , then  $G \neq Y_{k,k}$ ; and
- (iii) if  $k = 2$  then  $G$  is not a wheel.

Extending Theorem 1, Dirac and Erdős [3] showed that if a graph  $G$  has many more vertices of degree at least  $2k$  than vertices of lower degree, then  $G$  has  $k$  disjoint cycles.

**Theorem 3** ([3]). *If  $G$  is a simple graph and  $k \geq 3$ , and if the number of vertices in  $G$  with degree at least  $2k$  exceeds the number of vertices with degree at most  $2k - 2$  by at least  $k^2 + 2k - 4$ , then  $G$  contains  $k$  disjoint cycles.*

Dirac [2] described all 3-connected multigraphs that do not have two disjoint cycles and posed the following question:

**Question 4** ([2]). *Which  $(2k - 1)$ -connected multigraphs<sup>1</sup> do not have  $k$  disjoint cycles?*

We consider the class  $\mathcal{D}_k$  of multigraphs in which each vertex has at least  $2k - 1$  distinct neighbors. Our main result, Theorem 10, characterizes those multigraphs in  $\mathcal{D}_k$  that do not contain  $k$  disjoint cycles. Every  $(2k - 1)$ -connected multigraph is in  $\mathcal{D}_k$ , so this provides a complete answer to Question 4. Determining whether a multigraph is in  $\mathcal{D}_k$ , and determining whether a multigraph is  $(2k - 1)$ -connected, can be accomplished in polynomial time.

In the next section, we introduce notation, discuss existing results to be used later on, and state our main result, Theorem 10. In the last two sections, we prove Theorem 10.

## 2. PRELIMINARIES AND STATEMENT OF THE MAIN RESULT

**2.1. Notation.** For every multigraph  $G$ , let  $V_1 = V_1(G)$  be the set of vertices in  $G$  incident to loops. Let  $\tilde{G}$  denote the *underlying simple graph* of  $G$ , i.e. the simple graph on  $V(G)$  such that two vertices are adjacent in  $G$  if and only if they are adjacent in  $\tilde{G}$ . Let  $F = F(G)$

<sup>1</sup>Dirac used the word *graphs*, but in [2] this appears to mean *multigraphs*.

be the simple graph formed by the multiple edges in  $G - V_1$ ; that is, if  $G'$  is the subgraph of  $G - V_1$  induced by its multiple edges, then  $F = \widetilde{G'}$ . We will call the edges of  $F(G)$  *the strong edges of  $G$* , and define  $\alpha' = \alpha'(F)$  to be the size of a maximum matching in  $F$ . A set  $S = \{v_0, \dots, v_s\}$  of vertices in a graph  $H$  is a *superstar with center  $v_0$  in  $H$*  if  $N_H(v_i) = \{v_0\}$  for each  $1 \leq i \leq s$  and  $H - S$  has a perfect matching.

For  $v \in V$ , we define  $s(v) = |N(v)|$  to be the *simple degree* of  $v$ , and we say that  $\mathcal{S}(G) = \min\{s(v) : v \in V\}$  is the *minimum simple degree* of  $G$ . We define  $\mathcal{D}_k$  to be the family of multigraphs  $G$  with  $\mathcal{S}(G) \geq 2k - 1$ . By the definition of  $\mathcal{D}_k$ ,  $\alpha(G) \leq n - 2k + 1$  for every  $n$ -vertex  $G \in \mathcal{D}_k$ ; so we call  $G \in \mathcal{D}_k$  *extremal* if  $\alpha(G) = n - 2k + 1$ . A *big set* in an extremal  $G \in \mathcal{D}_k$  is an independent set of size  $\alpha(G)$ . If  $I$  is a big set in an extremal  $G \in \mathcal{D}_k$ , then since  $s(v) \geq 2k - 1$ , each  $v \in I$  is adjacent to each  $w \in V(G) - I$ . Thus

(2.1) every two big sets in any extremal  $G$  are disjoint.

**2.2. Preliminaries and main result.** Since every cycle in a simple graph has at least 3 vertices, the condition  $|G| \geq 3k$  is necessary in Theorem 1. However, it is not necessary for multigraphs, since loops and multiple edges form cycles with fewer than three vertices. Theorem 1 can easily be extended to multigraphs, although the statement is no longer as simple:

**Theorem 5.** *For  $k \in \mathbb{Z}^+$ , let  $G$  be a multigraph with  $\mathcal{S}(G) \geq 2k$ , and set  $F = F(G)$  and  $\alpha' = \alpha'(F)$ . Then  $G$  has no  $k$  disjoint cycles if and only if*

$$(2.2) \quad |V(G)| - |V_1(G)| - 2\alpha' < 3(k - |V_1| - \alpha'),$$

*i.e.,  $|V(G)| + 2|V_1| + \alpha' < 3k$ .*

*Proof.* If (2.2) holds, then  $G$  does not have enough vertices to contain  $k$  disjoint cycles. If (2.2) fails, then we choose  $|V_1|$  cycles of length one and  $\alpha'$  cycles of length two from  $V_1 \cup V(F)$ . By Theorem 1, the remaining (simple) graph contains  $k - |V_1| - \alpha'$  disjoint cycles.  $\square$

Theorem 5 yields the following.

**Corollary 6.** *Let  $G$  be a multigraph with  $\mathcal{S}(G) \geq 2k - 1$  for some integer  $k \geq 2$ , and set  $F = F(G)$  and  $\alpha' = \alpha'(F)$ . Suppose  $G$  contains at least one loop. Then  $G$  has no  $k$  disjoint cycles if and only if  $|V(G)| + 2|V_1| + \alpha' < 3k$ .*

Instead of the  $(2k - 1)$ -connected multigraphs of Question 4, we consider the wider family  $\mathcal{D}_k$ . Since acyclic graphs are exactly forests, Theorem 2 can be restated as follows:

**Theorem 7.** *For  $k \in \mathbb{Z}^+$ , let  $G$  be a simple graph in  $\mathcal{D}_k$ . Then  $G$  has no  $k$  disjoint cycles if and only if one of the following holds:*

- ( $\alpha$ )  $|G| \leq 3k - 1$ ;
- ( $\beta$ )  $k = 1$  and  $G$  is a forest with no isolated vertices;
- ( $\gamma$ )  $k = 2$  and  $G$  is a wheel;
- ( $\delta$ )  $\alpha(G) = n - 2k + 1$ ; or
- ( $\epsilon$ )  $k > 1$  is odd and  $G = Y_{k,k}$ .

Dirac [2] described all multigraphs in  $\mathcal{D}_2$  that do not have two disjoint cycles:

**Theorem 8** ([2]). *Let  $G$  be a 3-connected multigraph. Then  $G$  has no two disjoint cycles if and only if one of the following holds:*

- (A)  $\tilde{G} = K_4$  and the strong edges in  $G$  form either a star (possibly empty) or a 3-cycle;
- (B)  $G = K_5$ ;
- (C)  $\tilde{G} = K_5 - e$  and the strong edges in  $G$  are not incident to the ends of  $e$ ;
- (D)  $\tilde{G}$  is a wheel, where some spokes could be strong edges; or
- (E)  $G$  is obtained from  $K_{3,|G|-3}$  by adding non-loop edges between the vertices of the (first) 3-class.

Going further, Lovász [5] described *all* multigraphs with no two disjoint cycles. He observed that it suffices to describe such multigraphs with minimum (ordinary) degree at least 3, and proved the following:

**Theorem 9** ([5]). *Let  $G$  be a multigraph with  $\delta(G) \geq 3$ . Then  $G$  has no two disjoint cycles if and only if  $G$  is one of the following:*

- (1)  $K_5$ ;
- (2) A wheel, where some spokes could be strong edges;
- (3)  $K_{3,|G|-3}$  together with a loopless multigraph on the vertices of the (first) 3-class; or
- (4) a forest  $F$  and a vertex  $x$  with possibly some loops at  $x$  and some edges linking  $x$  to  $F$ .

By Corollary 6, in order to describe the multigraphs in  $\mathcal{D}_k$  not containing  $k$  disjoint cycles, it is enough to describe such multigraphs with no loops. Our main result is the following:

**Theorem 10.** *Let  $k \geq 2$  and  $n \geq k$  be integers. Let  $G$  be an  $n$ -vertex multigraph in  $\mathcal{D}_k$  with no loops. Set  $F = F(G)$ ,  $\alpha' = \alpha'(F)$ , and  $k' = k - \alpha'$ . Then  $G$  does not contain  $k$  disjoint cycles if and only if one of the following holds: (see Figure 2.1)*

- (a)  $n + \alpha' < 3k$ ;
- (b)  $|F| = 2\alpha'$  (i.e.,  $F$  has a perfect matching) and either
  - (i)  $k'$  is odd and  $G - F = Y_{k',k'}$ , or
  - (ii)  $k' = 2 < k$  and  $G - F$  is a wheel with 5 spokes;
- (c)  $G$  is extremal and either
  - (i) some big set is not incident to any strong edge, or
  - (ii) for some two distinct big sets  $I_j$  and  $I_{j'}$ , all strong edges intersecting  $I_j \cup I_{j'}$  have a common vertex outside of  $I_j \cup I_{j'}$ ;
- (d)  $n = 2\alpha' + 3k'$ ,  $k'$  is odd, and  $F$  has a superstar  $S = \{v_0, \dots, v_s\}$  with center  $v_0$  such that either
  - (i)  $G - (F - S + v_0) = Y_{k'+1,k'}$ , or
  - (ii)  $s = 2$ ,  $v_1v_2 \in E(G)$ ,  $G - F = Y_{k'-1,k'}$  and  $G$  has no edges between  $\{v_1, v_2\}$  and the set  $X_0$  in  $G - F$ ;
- (e)  $k = 2$  and  $G$  is a wheel, where some spokes could be strong edges;
- (f)  $k' = 2$ ,  $|F| = 2\alpha' + 1 = n - 5$ , and  $G - F = C_5$ .

The six infinite classes of multigraphs described in Theorem 10 are exactly the family of multigraphs in  $\mathcal{D}_k$  with no  $k$  disjoint cycles. So, the  $(2k - 1)$ -connected multigraphs with no  $k$  disjoint cycles are exactly the  $(2k - 1)$ -connected multigraphs that are in one of these classes. For any multigraph  $G$ , we can check in polynomial time whether  $G \in \mathcal{D}_k$  and

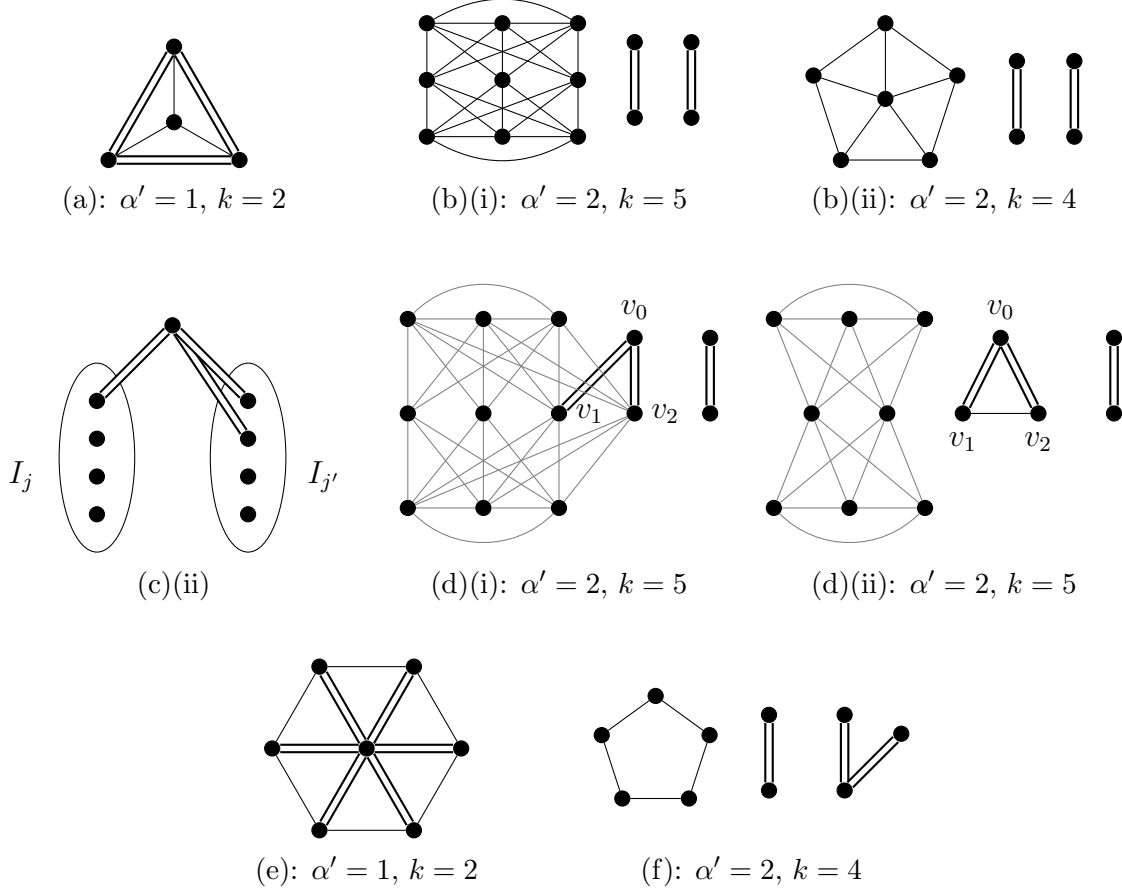


FIGURE 2.1. Examples of Subgraphs of Multigraphs Listed in Theorem 10

whether  $G$  is  $(2k - 1)$ -connected. If  $G \in \mathcal{D}_k$ , we can check in polynomial time whether any of the conditions (a)–(f) hold for  $G$ . Note that to determine the extremality of  $G$  we need only check whether  $G$  has an independent set of size  $n - 2k + 1$ . Such a set will be the complement of  $N(v)$  for some vertex  $v$  with  $s(v) = 2k - 1$ ; so all big sets can be found in polynomial time.

Note if  $G$  is  $(2k - 1)$ -connected, and (b)(i), (d)(i), or d(ii) holds, then  $k' \leq 1$ .

### 3. PROOF OF SUFFICIENCY IN THEOREM 10

Suppose  $G$  has a set  $\mathcal{C}$  of  $k$  disjoint cycles. Our task is to show that each of (a)–(f) fails. Theorem 9, case (2) implies (e) fails. Let  $M \subseteq \mathcal{C}$  be the set of strong edges (2-cycles) in  $\mathcal{C}$ ,  $h = |M|$ , and  $W = V(M)$ . Now  $h \leq \alpha'$ ; so  $n \geq 2h + 3(k - h) \geq 3k - \alpha'$ . Thus (a) fails. If  $n = 3k - \alpha'$  as in cases (b), (d) and (f), then  $h = \alpha'$  and  $G' = G - W$  is a simple graph of minimum degree at least  $2k' - 1$  with  $3k'$  vertices and  $k'$  cycles. By Theorem 2 all of (i)–(iii) hold for  $G'$ . In case (b),  $G' = G - F$ ; so (ii) and (iii) imply (b)(i) and (b)(ii) fail. In case (f),  $G' = G - (F - v) = v \vee C_5$  for some vertex  $v \in F$ . So (iii) implies (f) fails. In case (d),  $M$  consists of a strong perfect matching in  $F - S$  together with a strong edge  $v_0v \in S$ . If  $G - (F - S + v_0) = Y_{k'+1,k'}$  then either  $\alpha(G') = k' + 1$  or  $G' = Y_{k',k'}$ , contradicting (i) or (ii). So (d)(i) fails. Similarly, in case (d)(ii),  $G' \subseteq Y_{k',k'}$ , another contradiction.

In case (c),  $G$  is extremal. Every big set  $I$  satisfies  $|V(G) - I| < 2k$ . So some cycle  $C_I \in \mathcal{C}$  has at most one vertex in  $V(G) - I$ . Since  $I$  is independent,  $C_I$  has at most one vertex in  $I$ . Thus  $C_I$  is a strong edge and (c)(i) fails. Let  $J$  be another big set; then  $I \cap J = \emptyset$ . As cycles in  $\mathcal{C}$  are disjoint,  $C_I = C_J$  or  $C_I \cap C_J = \emptyset$ . Regardless,  $C_I \cap C_J \subseteq I \cup J$ . So (c)(ii) fails.

#### 4. PROOF OF NECESSITY IN THEOREM 10

Suppose  $G$  does not have  $k$  disjoint cycles. Our goal is to show that one of (a)–(f) holds. If  $k = 2$  then one of the cases (1)–(4) of Theorem 9 holds. If (1) holds then  $\alpha' = 0$ , and so (a) holds. Case (2) is (e). Case (3) yields (c)(i), where the partite set of size  $n - 3$  is the big set. As  $G \in \mathcal{D}_k$ , it has no vertex  $l$  with  $s(l) < 3$ . So (4) fails, because each leaf  $l$  of the forest satisfies  $s(l) \leq 2$ . Thus below we assume

$$(4.1) \quad k \geq 3.$$

Choose a maximum strong matching  $M \subseteq F$  with  $\alpha(G - W)$  minimum, where  $W = V(M)$ . Then  $|M| = \alpha'$ ,  $G' := G - W$  is simple, and  $\delta(G') \geq 2k - 1 - 2\alpha' = 2k' - 1$ . So  $G' \in \mathcal{D}_{k'}$ . Let  $n' := |V(G')| = n - 2\alpha'$ . Since  $G'$  has no  $k'$  disjoint cycles, Theorem 7 implies one of the following: (α)  $|G'| \leq 3k' - 1$ ; (β)  $k' = 1$  and  $G'$  is a forest with no isolated vertices; (γ)  $k' = 2$  and  $G'$  is a wheel; (δ)  $\alpha(G') = n' - 2k' + 1 = n - 2k + 1$ ; or (ε)  $k' > 1$  is odd and  $G' = Y_{k', k'}$ . If (α) holds then so does (a). So suppose  $n' \geq 3k'$ . In the following we may obtain a contradiction by showing  $G$  has  $k$  disjoint cycles.

**Case 1:** (β) holds. By (4.1), there are strong edges  $yz, y'z' \in M$ . As  $\mathcal{S}(G) \geq 2k - 1$ , each vertex  $v \in V(G')$  is adjacent to all but  $d_{G'}(v) - 1$  vertices of  $W$ .

*Case 1.1:*  $G'$  contains a path on four vertices, or  $G'$  contains at least two components. Let  $P = x_1 \dots x_t$  be a maximum path in  $G'$ . Then  $x_1$  is a leaf in  $G'$ , and either  $d_{G'}(x_2) = 2$  or  $x_2$  is adjacent to a leaf  $l \neq x_1$ . So  $vx_1x_2v$  or  $vx_1x_2lv$  is a cycle for all but at most one vertex  $v \in W$ . If  $t \geq 4$ , let  $s_1 = x_t$  and  $s_2 = x_{t-1}$ . Otherwise,  $G'$  is disconnected and every component is a star; in a component not containing  $P$ , let  $s_1$  be a leaf and let  $s_2$  be its neighbor. As before, for all but at most one vertex  $v' \in W$ , either  $v's_1s_2v'$  is a cycle or  $v's_1s_2l'v'$  is a cycle for some leaf  $l'$ . Thus  $G'[(V \setminus W) \cup \{u, v\}]$  contains two disjoint cycles for some  $uv \in \{yz, y'z'\}$ . These cycles and the  $\alpha' - 1$  strong edges of  $M - uv$  yield  $k$  disjoint cycles in  $G$ , a contradiction.

*Case 1.2:*  $G'$  is a star with center  $x_0$  and leaf set  $X = \{x_1, x_2, \dots, x_t\}$ . Since  $n' \geq 3k'$ ,  $t \geq 2$  and  $X$  is a big set in  $G$ . If (c)(i) fails then some vertex in  $X$ , say  $x_1$ , is incident to a strong edge, say  $x_1y$ . If  $t \geq 3$ , then  $G$  has  $k$  disjoint cycles:  $|M - yz + yx_1|$  strong edges and  $zx_2x_0x_3z$ . Else  $t = 2$ . Then  $n = 3\alpha' + 3k' = 2k + 1$ , as in (d); and each vertex of  $G$  is adjacent to all but at most one other vertices. If  $x_0z \in E(G)$  then again  $G$  has  $k$  disjoint cycles:  $|M - yz + yx_1|$  strong edges and  $zx_0x_2z$ , a contradiction. So  $N(x_0) = V(G) - z - x_0$ , and  $G[\{x_0, x_1, x_2, z\}] = C_4 = Y_{2,1}$ . Also  $y$  is the only possible strong neighbor of  $x_1$  or  $x_2$ : if  $u \in \{x_1, x_2\}$ ,  $y'z' \in M$  with  $y' \neq y$  (maybe  $y' = z$ ) and  $uy' \in E(F)$ , using the same argument as above, if  $z'x_0 \in E(G)$  then  $G$  has  $k$  disjoint cycles consisting of  $|M - y'z' + y'u|$  strong edges and  $G[G' - u + z']$ , a contradiction. Then  $x_0z' \notin E(G)$ , so  $z' = z$ , and  $y' = y$ . Thus  $S = N_F(y) \cap \{z, x_0, x_1, x_2\} + y$  is a superstar. So (d)(i) holds.

**Case 2:** (γ) holds. Then  $k' = 2$  and  $G'$  is a wheel with center  $x_0$  and rim  $x_1x_2 \dots x_tx_1$ . By (4.1), there exists  $yz \in M$ . Since (a) fails,  $t \geq 5$ . For  $i \in [t]$ ,

$$s(x_i) \geq 2k - 1 = 2\alpha' + 3 = 2\alpha' + |N(x_i) \cap G'|,$$

so  $x_i$  is adjacent to every vertex in  $W$ . If  $t \geq 6$ , then  $G'$  has  $k$  disjoint cycles:  $|M - yz|$  strong edges,  $yx_1x_2y$ ,  $zx_3x_4z$  and  $x_0x_5x_6x_0$ . Thus  $t = 5$ . If no vertex of  $G'$  is incident to a strong edge, then (b)(ii) holds. Therefore, we assume  $y$  has a strong edge to  $G'$ . The other endpoint of the strong edge could be in the outer cycle, or could be  $x_0$ . If some vertex in the outer cycle, say  $x_1$ , has a strong edge to  $y$ , then we have  $k$  disjoint cycles:  $|M - yz + yx_1|$  strong edges,  $zx_2x_3z$  and  $x_0x_4x_5x_0$ . The last possibility is that  $x_0$  has a strong edge to  $y$ , and (f) holds.

**Case 3:**  $(\epsilon)$  holds. Then  $k' > 1$  is odd,  $G' = Y_{k',k'}$  and  $n = 2\alpha' + 3k'$ . Let  $X_0 = \{x_1, \dots, x_{k'}\}$ ,  $X_1 = \{x'_1, \dots, x'_{k'}\}$ , and  $X_2 = \{x''_1, \dots, x''_{k'}\}$  be the sets from the definition of  $Y_{k',k'}$ . Observe

$$(4.2) \quad \overline{K}_{s+t} \vee (K_{2s} \cup K_{2t}) \text{ contains } s+t \text{ disjoint triangles.}$$

By degree conditions, each  $x' \in X_1 \cup X_2$  is adjacent to each  $v \in W$  and each  $x \in X_0$  is adjacent to all but at most one  $y \in W$ . If (b)(i) fails then some strong edge  $uy$  is incident with a vertex  $u \in V(G')$ . If possible, pick  $u \in X_1 \cup X_2$ . By symmetry we may assume  $u \notin X_2$ . Let  $yz$  be the edge of  $M$  incident to  $y$ . Set  $v_0 = y$  and  $\{v_1, \dots, v_s\} = V(F \cap G') + z$ . We will prove that  $\{v_0, \dots, v_s\}$  is a superstar, and use this to show that (d)(i) or (d)(ii) holds. Let  $G^* = G - (W - z)$ , and observe that  $Y_{k'+1,k'}$  is a spanning subgraph of  $G^*$  with equality if  $X_0 + z$  is independent.

Suppose  $xz \in E(G)$  for some  $x \in X_0 - u$ . Then  $G$  has  $k$  disjoint cycles:  $|M - yz + yu|$  strong edges,  $zxx'_1z$ , and  $k' - 1$  disjoint cycles in  $G^* - \{x, x'_1, u\}$ , obtained by applying (4.2) directly if  $u \in X_1$ , or by using  $T := x'_1x'_2x'_3x'_1$  and applying (4.2) to  $G^* - \{x, x'_1, u\} - T$  if  $u \in X_0$ . This contradiction implies  $zu$  is the only possible edge in  $G[X_0 + z]$ . Thus if  $y$  has two strong neighbors in  $X_0$  then  $X_0 + z$  is independent, and  $G^* = K_{k'+1,k'}$ . Also by degree conditions, every  $x \in X_0 - u$  is adjacent to every  $w \in W - z$ . So if  $y'z' \in M$  with  $y' \neq y$  and  $u' \in V(G')$ , then  $u'y' \notin E(F)$ : else  $x \in X_0 - u - u'$  satisfies  $xz' \in E(G)$  and  $xz' \notin E(G)$ . So  $\{v_0, \dots, v_s\}$  is a superstar. If  $X_0 + z$  is independent then (d)(i) holds; else (d)(ii) holds.

**Case 4:**  $(\delta)$  holds. Then  $\alpha(G') = n' - 2k' + 1 > n'/3$ , since  $n' \geq 3k'$ . So  $G'$  is extremal. Let  $J$  be a big set in  $G'$ . Then  $|J| = n' - 2k' + 1 = n - 2k + 1$ . So  $G$  is extremal and  $J$  is a big set in  $G$ . Also each  $x \in J$  is adjacent to every  $y \in V(G) - J$ . If (c)(i) fails then some  $x \in J$  has a strong neighbor  $y$ . Let  $yz$  be the edge in  $M$  containing  $y$ . In  $F$ , consider the maximum matching  $M' = M - yz + xy$ , and set  $G'' = G - V(M')$ . By the choice of  $M$ ,  $G''$  contains a big set  $J'$ , and  $J'$  is big in  $G$ . Since  $x \notin J'$ , (2.1) implies  $J' \cap J = \emptyset$  (possibly,  $z \in J'$ ). If (c)(ii) fails then there is a strong edge  $vw$  such that  $v \in J \cup J'$  and  $w \neq y$ . Moreover, by the symmetry between  $J$  and  $J'$ , we may assume  $v \in J'$ . Let  $uw$  be the edge in  $M$  containing  $w$ . Since  $M$  is maximum,  $u \neq z$ . Let  $M'' = M' - uw + vw$ . Again by the case,  $G - V(M'')$  contains a big set  $J''$ . Since  $x, v \notin J''$ ,  $J''$  is disjoint from  $J \cup J'$ . So  $n' \geq 3|J| > n'$ , a contradiction.  $\square$

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